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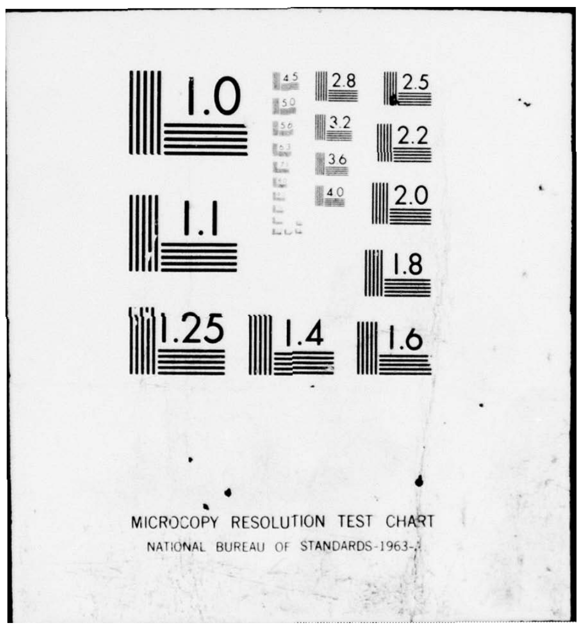
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A NONSTANDARD THEORY OF GAMES, PART IV, EQUILIBRIUM POINTS FOR *FINITE GAMES

10 Alain A. Lewis*

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Littauer #308
Cambridge, Mass. 02138

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*Presently at The Rand Corporation, Information Sciences Division, Santa Monica California.

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Introduction

In Part III of the series, the subject of interest was to obtain a non-atomic representation of *Finite noncooperative games in normal form. In the present paper we shall be concerned with features of the *Finite construction that allow a slight generalization of the solution concept employed, namely, that of an equilibrium point.

The present approach dispenses with the S-concavity requirement of the payoff functions by weakening that condition to require that the payoff functions be integrable over the product space of individual strategy sets. Each agent will be assumed to have a *Finite set of pure strategies, which is to say, that the cardinality of the set of pure strategies for each participant is always an integer $k \in N^*$. In this instance, it can be shown that the condition that the payoff functions be integrable follows from the continuity of the payoff functions in the Q-discrete topology of pointwise convergence on the space $T = \prod_{j \in F^*} [(S_j, 2^{S_j})]$, where S_j is the *Finite set of pure strategies available to the j^{th} participant. This comprises Theorem II.3. Given the integrability of the payoff functions, the results of Anderson [1], Loeb [3] and Brown [6] are then combined with an analog of the Schauder-Tychonoff Fixed Point Theorem to obtain the principal result, Theorem I.5.

Our notion of quasi-finitely determined appears explicitly in Peleg [5] and covertly in Wesley [4], Theorem 4.8. However, the origin of integrals generated by strategies appears to be in the work of Dubins and Savage [7], Ch. I.6, and the notion of quasi-finitely determined payoff functions in the context of integration seems to be attributable in origin to the work of Kalmár [8]. The suggestion that Kalmár's result can be given a topological characterization Dubins and Savage attribute to John Myhill, op. cit., Theorem 1 of Ch. I.7 and Theorem 5 of Ch. I.8. The non-standard characterization is, of course, ours.

It should be remarked that the framework we employ is strictly weaker than that of Peleg's in the following sense. Peleg's use of the Axiom of Choice in the strong version of Tychonoff's theorem on product spaces of nondenumerable cardinality is at odds with the principal desideratum of his result. For if we select a measure space as the set of players, of nondenumerable cardinality, the characterization of an equilibrium point as a property satisfied for all players, and not just on sets that differ by a null set of measure, as is well-known, is in contradiction to the Axiom of Choice. Peleg's original result must be interpreted then, for the case of a denumerably infinite set of players only.

On the other hand, our construction utilizes a weaker assumption, embodied in the construction of $\aleph_1$-saturated enlargements, namely, the Boolean Prime Ideal Theorem, which

is, in fact, independent of the Axiom of Choice. In addition, because of the formal properties of *Finite sets, namely, that any internal *Finite set of the form $[0,\omega]$, for $\omega \in N^* - N$ has external cardinality of at least $2^{\aleph_0}$, the nonstandard measure theoretic formulation achieves Peleg's original goal under weaker assumptions.

I The *Finite Framework and Existence Proofs

Let M^* be an $\aleph_1$-saturated enlargement and consider the internal *Finite set, $F^* = [0,\omega]$, for $\omega \in N^* - N$, which we will assume is the set of players for the game. For each $j \in F^*$, let there then be associated a *Finite set $T_j = [0,n]$, for $n \in N^* - N$, such that T_j is internal. We will assume the T_j to be the pure strategy sets of the participants, and for the sake of simplicity, we will also assume that all agents have the same number of strategies available to them, so that $||T_j|| = n$ for a fixed $n \in N^* - N$ for every $j \in F^*$.

By a mixed strategy for the j^{th} player we will mean an internal assignment $m_j : T_j \to R^*_+$, such that $\sum_{s_j^i \in T_j} m_j(s_j^i) = 1$. The set of all mixed strategies for each player $j \in F^*$, will be denoted as Σ^j. Then, obviously, it is permissible to consider Σ^j as an internal *Finite simplex of dimension n for each $j \in F^*$. The product space $\prod_{j \in F^*} [(\Sigma^j)]$ will be denoted as $\bar{\Sigma}$.

Let $A(T_j)$ be the algebra of internal subsets of T_j. Then each $m_j \in \Sigma^j$ induces a probability on $A(T_j)$ as follows. $P_j(S) = (f \circ m_j)(S)$ for $S \in A(T_j)$, where $(f \circ m_j)(S)$ has the form $\sum_{s_j^i \in S} m_j(s_j^i)$. It is then permissible to denote as $(T_j, A(T_j))$ the internal *Finite measure space, normalized in probability, on the internal subsets of $A(T_j)$. Let $(T, \mathcal{B})$ denote the product measure $\prod_{j \in F^*} [T_j, A(T_j)]$ and note that the internality of $(T, \mathcal{B})$ follows from the internality of the

component spaces $(T_j, A(T_j))$ and the internality of the set of players F^*.

Definition I.1: Let X be an internal *Finite set and A(X) the collection of internal subsets of X. Then $(X, A(X), u_X)$ for $u_X(S) = \frac{\|X \cap S\|}{\|X\|}$ if $S \in A(X)$, is an internal *Finite measure space.

A function $f : X \to R^*$ is finite if

(i) f is u_X-measurable

(ii) for some standard $n \in N$, $|f(x)| < n$ for any $x \in X$

(iii) $st\left(u_X(\{x : f(x) \neq 0\})\right) < \infty$

Definition I.2: A function $f : X \to R^*$ is said to be S-integrable if·

(i) f is u_X-measurable

(ii) $st(\int_X |f| du_X) < \infty$

(iii) $S \in A(X)$ and $(u_X(S) = 0)_{\text{Mod } M_1}$, then

$(\int_S |f| du_X = 0)_{\text{Mod } M_1}$

(iv) $S \in A(X)$ and $(f(S) = 0)_{\text{Mod } M_1}$, then

$(\int_S |f| du_X = 0)_{\text{Mod } M_1}$

For each player $j \in F^*$, let there be defined a function $H_j : (f \circ \bar{\Sigma}) \to R^*_+$ such that H_j is finite for $\bar{m} \in f \circ \bar{\Sigma}$ on the product measure space (T, B). The domain of H_j for each $j \in F^*$ are vectors of the form $\bar{m} = (f \circ m_1, \ldots, f \circ m_\omega)$ for some $m \in \bar{\Sigma}$.

Then the *Finite game in normal form, as considered in this context, is given as:

$$N^*_{F^*} = \langle \{T_j\}_{j\epsilon F^*}, F^*, \{H_j\}_{j\epsilon F^*} \rangle$$

Let $j \epsilon F^*$, $\tilde{z}_j = (f \circ z_j)$ for $z_j \epsilon \Sigma^j$ and $\tilde{m} = (f \circ m)$ for $m \epsilon \tilde{\Sigma}$, and define $\tilde{n} = \tilde{m}|z_j$ for $\tilde{n} = (f \circ n)$ for $n \epsilon \tilde{\Sigma}$ to be $\tilde{n}_k = \tilde{m}_k$ for $k \neq j$ and $\tilde{n}_j = \tilde{z}_j$.

Definition I.3: A vector $\tilde{m}^0 \epsilon \tilde{\Sigma}$ is an equilibrium point of the game $N^*_{F^*}$ if for the set of players, F^*,

$$(\forall m_j \epsilon \Sigma^j) \left(\int_T H_j(\tilde{m}^0) d\tilde{m}^0 \geq \int_T H_j(\tilde{m}^0|\tilde{m}_j) d\tilde{m}^0|\tilde{m}_j \right)$$

for a.e. $j \epsilon F^*$.

By a.e. in F^* is meant that the set of players for which the property is false is negligible, i.e., $S \epsilon A(F^*)$ such that $\left(\frac{||S||}{||F^*||} = 0 \right)_{\text{Mod } M_1}$.

Definition I.4: A function $\psi : \tilde{\Sigma} \to R^*$ is said to be finitely determined on $\tilde{\Sigma}$ if there is an internal standard finite set $A \subseteq F^*$, such that for $m_1, m_2 \epsilon \tilde{\Sigma}$, $m_{1j} = m_{2j}$ for $j \epsilon A$, implies that $\psi(m_1) = \psi(m_2)$.

A function $\psi : \tilde{\Sigma} \to R^*$ is quasi-finitely determined if for every standard epsilon, $\epsilon > 0$, there exists a finitely determined function, $\psi_{(\epsilon)}$, such that $|\psi_{(\epsilon)}(m) - \psi(m)| < \epsilon$ for any $m \epsilon \tilde{\Sigma}$. Alternatively, ψ is quasi-finitely determined if it is the uniform F-limit of a sequence of finitely determined functions on $\tilde{\Sigma}$, $\{\psi_k\}_{k\epsilon N}$.

Theorem I.5: If the payoff functions are quasi-finitely determined, then the game $N^*_{F^*}$ has an equilibrium point.

Proof: By the assumption, each H_j for $j \in F^*$ is such that there exists a sequence, $\{H_{jn}\}_{n \in N}$, of finitely determined functions converging uniformly on $\tilde{\Sigma}$ to H_j. For any H_{jn} in the sequence, and $m \in \tilde{\Sigma}$, the S-bounded character of H_{jn}, and the Q-compactness of $\tilde{\Sigma}$, by virtue of each Σ^j being Q-compact, yields that H_{jn} is finite in the sense of Definition I.1. The integral of H_{jn} on T for $\tilde{m} \in (f \circ \tilde{\Sigma})$ is given as $\int_T H_{jn}(\tilde{m})d\tilde{m}$ and is S-continuous in the weak S-topology, which by virtue of the internality of F*, coincides with the product S-topology on $\prod_{j \in F^*} [(f \circ \Sigma^j)]$. We will employ the following result.

Lemma I.5.1: Let $f : T \to R^*$ be u_T-measurable. Then f is S-integrable if and only if there exists a sequence of finite functions $\{f_n\}_{n \in N}$ such that $\underset{n \to \infty}{st}(\int_T |f - f_n| du_T) = 0$.

Proof: Anderson [1], Theorem 4.

Since any $\tilde{m} \in (f \circ \tilde{\Sigma})$ induces an internal *Finite measure on T, for each $j \in F^*$, $H_j(\tilde{m})$ is T-measurable with respect to the measure on T induced by $\tilde{m}$. Further, since each H_j, for $j \in F^*$, is quasi-finitely determined, by Lemma I.5.1, given any $\tilde{m} \in (f \circ \tilde{\Sigma})$, each H_j is S-integrable. Since the sequence

of integrals $\{\int_T H_{jn}(\tilde{m})d\tilde{m}\}_{n\varepsilon N}$ is comprised of S-continuous members and converges uniformly to $\int_T H_j(\tilde{m})d\tilde{m}$, $\int_T H_j(\tilde{m})d\tilde{m}$ is S-continuous on $(f \circ \tilde{\Sigma})$.

Let the function $\rho(\tilde{m},\tilde{n})$ be defined on $(f \circ \tilde{\Sigma}) \times (f \circ \tilde{\Sigma})$ to be $\rho(\tilde{m},\tilde{n}) = \sum_{j\varepsilon F^*}\left[\int_T H_j(\tilde{m}|\tilde{n}_j)d\tilde{m}|\tilde{n}_j\right]$. For a fixed $\tilde{m}\,\varepsilon\,(f \circ \tilde{\Sigma})$, consider the mapping $\Gamma(m) = \{\tilde{z}\,\varepsilon\,(f \circ \tilde{\Sigma}) : \rho(\tilde{m},\tilde{z}) = \max_{\tilde{n}\varepsilon(f\circ\tilde{\Sigma})} \rho(\tilde{m},\tilde{n})\}$ where the max is with respect to the inequality "$\geq$".

By the results of Loeb [2], Theorem 5.1, and Loeb [3], the range of $\int_T H_j(\tilde{m})d\tilde{m}$ is S-convex for any $\tilde{m}\,\varepsilon\,(f \circ \tilde{\Sigma})$, where $\int_T H_j(\tilde{m})d\tilde{m}$ can be regarded as an infinitesimal vector measure on T, v(T).

Lemma I.5.2: $\Gamma(f \circ \tilde{\Sigma})$ is S-convex.

Proof: We follow Brown [6], Theorem 6, in analog.

Suppose $\tilde{z}$ and $\tilde{y}$ are distinct in $(f \circ \tilde{\Sigma})$. Then for some $\tilde{m}\,\varepsilon\,(f \circ \tilde{\Sigma})$, $\rho(\tilde{m},\tilde{z})$ and $\rho(\tilde{m},\tilde{y})$ define measures $v^0(A) = \sum_{j\varepsilon F^*}\left[\int_A H_j(\tilde{m}|\tilde{z}_j)d\tilde{m}|\tilde{z}_j\right]$ and $\bar{v}^0(A) = \sum_{j\varepsilon F^*}\left[\int_A H_j(\tilde{m}|\tilde{y}_j)d\tilde{m}|\tilde{y}_j\right]$, for $A\,\varepsilon\,\mathcal{B}$. Let us now form the vector measure $v(A) = \langle v^0(A), \bar{v}^0(A)\rangle$ $A\,\varepsilon\,\mathcal{B}$. Then, by the principal result of Loeb [3], for any $\lambda\,\varepsilon\,(0,1)$, there is some S in $\mathcal{B}$, such that

$$\left[v(S) = \lambda\big(v(T)\big)\right]_{\text{Mod } M_1}$$

Then,

$$\left[v(T-S) = (1-\lambda)\big(v(T)\big)\right]_{\text{Mod } M_1}$$

Allow

$$v'(A) = \begin{cases} v^0(A) & \text{for } A \subseteq S \\ \bar{v}^0(A) & \text{for } A \subseteq T - S \end{cases}$$

Then, $v'(T) = v^0(S) + \bar{v}^0(T - S)$. However,

$$\lambda\big(v(T)\big) = <\lambda(v^0(T)),\lambda\big(\bar{v}^0(T)\big)>$$

and

$$(1 - \lambda)\big(v(T)\big) = <(1 - \lambda)\big(v^0(T)\big),(1 - \lambda)\big(\bar{v}^0(T)\big)>$$

Then it follows that

$$\Big[\big(v^0(S) + \bar{v}^0(T - S)\big) = \lambda\big(v^0(T)\big) + (1 - \lambda)\bar{v}^0(T)\Big]_{\text{Mod } M_1}$$

Then,

$$v'(T) = \sum_{j\varepsilon F^*}\Big(\int_T H_j(\tilde{m}|\tilde{x}_j)d\tilde{m}|\tilde{x}_j\Big)$$

for

$$\tilde{x}_j = \begin{cases} \tilde{z}_j & j \varepsilon S \\ \tilde{y}_j & j \varepsilon T - S \end{cases}$$

Therefore, for some $\tilde{x} \varepsilon (f \circ \tilde{\Sigma})$, it follows that

$$\Big[\Big(\sum_{j\varepsilon F^*}\big(\int_T H_j(\tilde{m}|\tilde{x}_j)d\tilde{m}|\tilde{x}_j\big)\Big) = \lambda\big(v^0(T)\big) + (1-\lambda)\bar{v}^0(T)\Big]_{\text{Mod } M_1}$$

and therefore from the fact that

$$\Big[\lambda\big(v^0(T)\big)+(1-\lambda)\bar{v}(T) = \lambda\Big(\sum_{j\varepsilon F^*}\int_T H_j(\tilde{m}|\tilde{z}_j)d\tilde{m}|z_j\Big)+(1-\lambda)\Big(\sum_{j\varepsilon F^*}\int_T H_j(\tilde{m}|\tilde{y}_j)d\tilde{m}|y_j\Big)\Big]_{\text{Mo}}$$

the lemma follows easily.

Q.E.D.

From the internality of F*, and since each integral is S-continuous in $\tilde{m}$, $\rho(\tilde{m},\tilde{n})$ is seen to be S-continuous in $\tilde{n}$. Since the Q-topology refines the S-topology, $\rho(\tilde{m},\tilde{n})$ is

a fortiori Q-continuous in $\tilde{n}$. The upper Q-semicontinuity of Γ on $(f \circ \tilde{\Sigma})$ is then easily verified. Then the auxiliary mapping $\psi(\tilde{m}) = \{\tilde{x} \in (f \circ \tilde{\Sigma}) : \tilde{x} \in \text{Q-con}\,\Gamma(\tilde{m})\}$ has a fixed point along the lines of the appropriate analog of the Schauder-Tychonoff Fixed Point Theorem. Then there is some vector $\tilde{m}^0 \in (f \circ \tilde{\Sigma})$ such that $\tilde{m}^0 \in \text{Q-con}\big(\Gamma(\tilde{m}^0)\big)$. Then by Lemma 1, p. 6 of Brown [6], there is some $\tilde{m}^* \in (f \circ \tilde{\Sigma})$ such that $(\tilde{m}^* = \tilde{m}^0)_{\text{Mod } M_1}$. We claim that $\tilde{m}^*$ is an equilibrium point in the sense of Definition I.3. For, if $\tilde{m}^*$ is not, then for some $\tilde{z} \in (f \circ \tilde{\Sigma})$ $\rho(\tilde{m}^*, \tilde{z}) \gg \rho(\tilde{m}^*, \tilde{m}^*)$. However, the Q-continuity of $\rho(\tilde{m}, \tilde{n})$ on $(f \circ \tilde{\Sigma})$ and the fact that $(\tilde{m}^* = \tilde{m}^0)_{\text{Mod } M_1}$, would then imply that $\rho(\tilde{m}^*, \tilde{z}) \gg \rho(\tilde{m}^0, \tilde{m}^0)$, which is clearly false. Then for any $\tilde{z} \in (f \circ \tilde{\Sigma})$, $\rho(\tilde{m}^*, \tilde{m}^*) \geq \rho(\tilde{m}^*, \tilde{z})$. That $\tilde{m}^*$ is S-maximal a.e. in F^* is seen to follow from the fact that $\big(\rho(\tilde{m}^*, \tilde{m}^*) = \rho(\tilde{m}^0, \tilde{m}^0)\big)_{\text{Mod } M_1}$.

Q.E.D.

II. Sufficient Conditions for Quasi-Finitely Determined Payoff Functions

Let $(T,\mathcal{B})$ be endowed with the Q-topology of convergence on *Finite sets denoted as $Q\text{-}D_T$. The closed sets of $Q\text{-}D_T$ are pointwise closed in the product algebra $\mathcal{B} = \prod_{j\varepsilon F^*} [A(T_j)]$. The standard reference for such a topology is Kelly's General Topology, Ch. 7; the formal properties there are identical to those of the Q-topological context in M*.

Consider next the quotient topology on $Q\text{-}D_T$, $Q\text{-}D_T(\text{Mod } N^* \cap M_1)$, where $N^* \cap M_1$ are the finite members of N*. Points of $(T,\mathcal{B})$ under $Q\text{-}D_T(\text{Mod } N^* \cap M_1)$ are actually equivalence classes mod some fixed standard natural number.

Definition II.1: Let us denote as $\mathcal{B}^*(S)$ the space of S-bounded R*-valued functions in M* defined in internal sets with norm given as $|f| = \sup_{s\varepsilon S}|f(s)|$. A set A in $\mathcal{B}^*(S)$ is said to distinguish points of S if for $s,t \,\varepsilon\, S$, such that $s \neq t$, then there is some $f \,\varepsilon\, A$ such that $f(s) \neq f(t)$.

Clearly, then, the internal algebra of all S-bounded finitely determined functions on $(T,\mathcal{B})$ distinguishes points of T in the topology $Q\text{-}D_T(\text{Mod } N^* \cap M_1)$. A fortiori, the internal algebra of S-bounded finitely determined functions renders $\tilde{\Sigma}$ distinguishable as well. Let us denote as $FD(\tilde{\Sigma})$, the space of all S-bounded finitely determined functions on $\tilde{\Sigma}$. The following result is Stone's well-known generalization of Weirstrass's theorem, which we adapt to the Q-context in M*.

Lemma II.2: Let S be a Q-compact Hausdorff space and let C(S) denote the internal algebra of all R*-valued continuous functions on S. Let $\mathfrak{A}$ be a closed sub-algebra of C(S) containing the identity mapping. Then $\mathfrak{A} = C(S)$ if and only if $\mathfrak{A}$ distinguishes points on S.

We give next a theorem which serves to sharply characterize those payoff functions that are quasi-finitely determined. It would seem to be a useful result for characterizing *Finite games in other contexts as well, especially in the area of those games derived from Nonstandard Exchange Economies.

Theorem II.3: For the game $N^*_{F^*} = \langle \{T_j\}_{j\varepsilon F^*}, F^*, \{H_j\}_{j\varepsilon F^*} \rangle$, if each H_j is continuous on $(T,\mathcal{B})$ in the topology $Q - D_T(\text{Mod } N^* \cap M_1)$, then the payoff functions are quasi-finitely determined. Conversely, if H_j is quasi-finitely determined, then H_j is continuous on $(T,\mathcal{B})$ in the topology $Q - D_T(\text{Mod } N^* \cap M_1)$.

Proof: Clearly, the topology $Q - D_T(\text{Mod } N^* \cap M_1)$ is Hausdorff as applied to $(T,\mathcal{B})$ since each $(T_j, A(T_j))$ is Hausdorff under $Q - D_T(\text{Mod } N^* \cap M_1)$. The internal sets of $\mathcal{B}(\text{Mod } N^* \cap M_1)$ form a base for this topology. Clearly, any cover of T by members of $\mathcal{B}$ has a *Finite subcover. Then T, and therefore $\tilde{\Sigma}$, is Q-compact and Hausdorff.

Then in Lemma I.7, allow $S = \tilde{\Sigma}$ and $\mathfrak{U} = FD(\tilde{\Sigma})$ to obtain $FD(\tilde{\Sigma}) = C(\tilde{\Sigma})$. It is plain to see that any $\psi \in C(\tilde{\Sigma})$ taking nonnegative values in R^*_+ is such that $\psi \in cl(C(\tilde{\Sigma}))$. Then by reasoning totally analogous to Kelly, op. cit., Section R, p. 244, since the $Q - D_T$ topology has identical formal features as its standard counterpart, for any $f \in C(\tilde{\Sigma})$ and $m \in \tilde{\Sigma}$, there is a $\psi \in cl(C(\tilde{\Sigma}))$ such that $\psi(m) = f(m)$ and for any $\varepsilon > 0$, ε positive and standard, $|f(n) - \psi(n)| < \varepsilon$ for all $n \in \tilde{\Sigma}$. In particular, since $FD(\tilde{\Sigma}) = C(\tilde{\Sigma})$, this last statement is true for $H_j \in FD(\tilde{\Sigma})$, which establishes the theorem.

Q.E.D.

Appendix

We have thought it useful to include a few salient aspects of the Theory of Nonstandard Integration as developed by Loeb and Bernstein in "A Nonstandard Integration Theory for Unbounded Functions," in The Victoria Symposium on Nonstandard Analysis, A. Hurd and P. Loeb, editors, Springer Verlag, 1974.

Let $(X,\mathcal{B},u)$ be a standard σ-finite measure space with u defined on $\mathcal{B}$, the Borel sets of X. The principal aim of the Loeb-Bernstein construction is to demonstrate that, within the context of the enlargement, M^*, one can restrict integration to appropriately defined *Finite subsets $y \subseteq X^*$, and obtain in close approximation, the proper values of the integrals of all the standard positive u-measurable functions.

Let us assume that X has the form $X = \left(\bigcup_{n=1}^{\infty} X_n\right)$ for $\{X_n\}_{n\varepsilon N} \subseteq X$ such that $u(X_n) < \infty$ and $X_n \subseteq X_{n+1}$. Let M_{R^+} be the nonnegative real valued functions on $(X,\mathcal{B})$, and let L^+ be the u-integrable functions in M_{R^+} where by convention, we denote $M^\infty = M_{R^+} - L^+$. Let M_{F^*} be a *Finite subset of $M^*_{R^+}$ such that if $L \varepsilon M_{R^+}$ then $L \varepsilon M^*_{R^+}$.

For a partition $P \varepsilon P^*$, indexed by $I \subseteq N^*$, and some set C in the enlargement M^*, let $I(C) = \{i \varepsilon I : A_i \varepsilon P. \wedge . A_i \subseteq C\}$ for A_i such that $u(A_i) > 0$. Then $||I(C)||$ will denote the number of elements in $I(C)$ with values in N^*. Specifically, $||\cdot|| : I(C) \xrightarrow[1-1]{} F^*$ for F^* internal in N^*.

Theorem A.1: For a given choice of $e_1 \in M_1^+$, and $\omega^0 \in N^* - N$, there exists integers n and $\mathcal{B}$ in $N^* - N$ such that:

(i) If X_n is the set corresponding to n in the collection $\{X_n\}^*$, then there is an internal set $Y \subseteq X_n$ with the property $u^*(Y) > u^*(X_n) - e_1$

(ii) For any $f \in L^+$, $\int_Y f^*du^* > \int_X fdu^* - e_2$

(iii) For any $g \in M^\infty$, $\int_Y g^*du^* > \omega^0$

(iv) For any $h \in M_{F^*}$, $\sup\limits_Y h \leq \mathcal{B}$

Proof: Loeb and Bernstein, op. cit., Theorem 1.

Q.E.D.

Theorem A.2: Given e_1, ω^0, n, $\mathcal{B}$ and Y as in Theorem A.1, there is a partition $P_0 \in P_0^*$ such that Y is exactly the union of sets from P_0 and for any partition $P \geq P_0$ in P^*, one has, in terms of the index set I for P, and an arbitrary choice of points $X_i \in A_i$, $i \in I(Y)$,

(i) $\left| \int_B fdu - \sum\limits_{i \in I(Y \cap B^*)} f^*(x_i)u^*(A_i) \right| < 2e_2$

for any $f \in L^+$ and $B \in \mathcal{B}$

(ii) $\sum\limits_{i \in I(Y)} g^*(x_i)u^*(A_i) > \omega^0 - e_1$

Proof: By Theorem A.1, each $h \in M_{F^*}$ is bounded by $\mathcal{B}$ on Y. Therefore for some $P_0 \in P_0^*$, P_0 is a partition such that $P_0 \geq \{Y, X^* - Y\}$ and for any set $C \in P_0$ for which $C \subseteq Y$, and each $h \in M_{F^*}$, one has $\sup\limits_C h - \inf\limits_C h < \frac{e_1}{u^*(Y)}$.

Select any partition $P \geq P_0$ with an index set I and choose a point $x_i \in A_i$ for each $i \in I(Y)$. Given any $h \in M_{F^*}$, one then has

$$\int_Y h du^* = \sum_{i \in I(Y)} \left(h(x_i) + \delta_i\right) u^*(A_i)$$

for $|\delta_i| < e_1/u^*(Y)$ for $i \in I(Y)$. Then,

$$\left|\int_Y h du^* - \sum_{i \in I(Y)} h(x_i) u^*(A_i)\right| < \frac{e_1}{u^*(Y)} \sum_{i \in I(Y)} u^*(A_i) = e_1$$

If $f \in L^+$, and $B \in \mathcal{B}$, one can allow the function $h = f^* \cdot \chi_{B^*}$ in the above.

This is Theorem 2 of Loeb and Bernstein, op. cit.

Q.E.D.

To explain the use of partitions in the above, let P be the collection of all finite u-measurable partitions of X. Then if $P_\alpha \in P$, then $P_\alpha = \{B_1, B_2, \dots, B_n\}$ for $X = \dot{\bigcup}_{i=1}^{n} B_i$, and for $1 \leq i \leq j \leq n$, $B_i \in \mathcal{B}$, $B_i \neq \phi$, $B_i \cap B_j = \phi$ if $i \neq j$. One symbolizes $P_{\mathcal{B}} \geq P_\alpha$ for $P_\alpha, P_{\mathcal{B}} \in P$ if any B in P_α is such that $B = {}_{C \in P_{\mathcal{B}}} C$ for $C \subset B$. If $P_{\mathcal{B}} \geq P_\alpha$, then we say that $P_{\mathcal{B}}$ is a refinement of P_α. The relation $\geq$ is concurrent on members of P, which is to say that for a finite chain, $\{(P_{\alpha j}, P_{\mathcal{B} j})\}_{j=1}^{K}$, defined on $\geq$ there is a "top," $\hat{P}$, that bounds it. Then because M* is an N_1-saturated enlargement, for some P_0 in P* $P_0 \geq P_\alpha$ for any $P_\alpha \in P$. Then one symbolizes $P \in P_0^*$ if $P \geq P_\alpha$ for each $P_\alpha \in P$.

The following two corollaries serve to indicate our earlier use of the nonstandard integral in the section on *Finite framework and existence proofs over the *Finite set T.

Their interpretation in terms of the concepts employed in the referenced section, namely, $H_j \in FD(\tilde{\Sigma})$ and $\tilde{m} \in (f \circ \tilde{\Sigma})$, should be clear.

Corollary: If u is a non-atomic measure, one can choose the partition $P \geq P_0$ with index set I, so that for any choice if $x_i \in A_i$, $i \in I(Y)$, and $f \in L^+$, $g \in M^\infty$, and $B \in \mathcal{B}$, one obtains:

$$\text{(i)} \quad \left| \int_B f du - \frac{u^*(Y)}{||I(Y)||} \cdot \sum_{i \in I(Y \cap B^*)} f^*(x_i) \right| < 3e_1$$

$$\text{(ii)} \quad \frac{u^*(Y)}{||I(Y)||} \sum_{i \in I(Y)} g^*(x_i) > \omega^0 - 2e_1$$

Proof: One recalls that $\mathcal{B}$ is an upper bound for the functions in M_{F^*} on Y. One can then find a partition $P \geq P_0$ indexed by I such that for each $i \in I(Y)$

$$\left| u^*(A_i) - \frac{u^*(Y)}{I(Y)} \right| < \frac{e_1}{||I(Y)|| \cdot \mathcal{B}}$$

Therefore, given $h \in M_{F^*}$, and $B \in \mathcal{B}$, one has

$$\left| \sum_{i \in I(Y \cap B^*)} h(x_i) u^*(A_i) - \frac{u^*(Y)}{||I(Y)||} \cdot \sum_{i \in I(Y \cap B^*)} h(x_i) \right|$$

$$\leq \sum_{i \in I(Y \cap B^*)} h(x_i) \left| u^*(A_i) - \frac{u^*(Y)}{||I(Y)||} \right|$$

$$< \frac{e_1}{||I(Y)|| \cdot \mathcal{B}} \sum_{i \in I(Y) \cap B^*)} h(x_i) \leq e_1$$

Q.E.D.

Corollary: If $u(X) = 1$, then for any $f \varepsilon L^+$, $g \varepsilon M^\infty$ and $B \varepsilon \mathcal{B}$,

$$\left(\int_B f du = \int_{B^*} f^* du^*\right)_{\text{Mod } M_1}$$

where

$$\int_{B^*} f^* du^* = \frac{1}{\|I(X^*)\|} \sum_{i \varepsilon I(B^*)} f^*(x_i)$$

Then, in the terminology of the earlier referenced section, let $u^*(T) = \frac{\|T\|}{\|T\|} = \tilde{m}(T) = 1$ for $\tilde{m} \varepsilon (f \circ \tilde{\Sigma})$. Then for a payoff function, $H_j \varepsilon FD(\tilde{\Sigma})$, one has the following representation for the integral of H_j on T:

$$\int_T H_j(\tilde{m}) d\tilde{m} = \frac{1}{\|T\|} \sum_{s^i \varepsilon T} H_j\left(m(s^i)\right)$$

To see how one might generate *Finite sets of pure strategies from a standard metric space, taking as given a *Finite set of participants, consider the following.

If one starts by requiring that each $j \varepsilon F^*$ has a strategy space of the form of, say, the unit interval, $[0,1]$, one can form in M^* a covering of $[0,1]^*$ by the infinitesimal "boxes" $\left[\frac{K}{\omega}, \frac{K+1}{\omega}\right]$ for $K \varepsilon N^*$, $\omega \varepsilon N^*$ and $K < \omega$. Since M^* enlarges M, $[0,1] \subsetneq [0,1]^*$ and the "boxes" cover $[0,1]$ as well. Let the equivalence relation, $x \sim y$, for $x,y \varepsilon [0,1]$, be defined to hold if and only if $|x^* - y^*| \leq 1/\omega$, or alternatively phrased, if and only if x and y are in the same "box," the length of any box being $1/\omega$. Then we may regard the $\left[\frac{K}{\omega}, \frac{K+1}{\omega}\right]$ as equivalence classes of the form $[x] = \{y^* \varepsilon [0,1]^* : x^* \sim y^*\}$ for $x \varepsilon [0,1]$. It is plain that

the number of boxes equals the number of equivalence classes and that number is *Finite, namely, $\omega \in N^* - N$.

If one began with a simplex $[0,1]^n$ for $n \in N$, then the appropriate procedure would involve a *Finite number of infinitesimal "cubes," $\left[\frac{K}{\omega}, \frac{K+1}{\omega}\right]^n$.

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13. ABSTRACT

The results obtained in Theorem I.1.15 of Part III of the series on the existence of Nash-type equilibria for Non-cooperative *Finite Games are considered under weaker assumptions. The assumption that the payoff functions be S-concave is dropped in favor of the condition that they be quasi-finitely determined, a condition attributable in origin to the work of Kalmár[8]. It is then shown, making use of the results of Anderson [1] and Loeb [2,3], that the latter assumption implies the integrability of the payoff functions in the product measure space induced by the mixed strategies of the players. Necessary and sufficient conditions for the payoff functions to be quasi-finitely determined are given a topological characterization.

*Presently at The Rand Corporation, Information Sciences Division, Santa Monica, California.

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